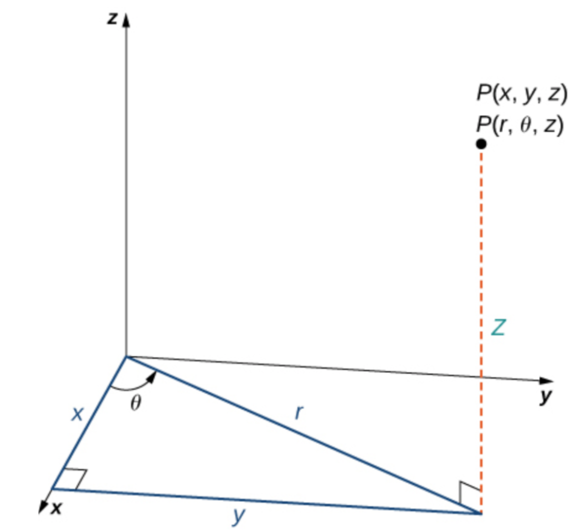
SECTION 16.5: TRIPLE INTEGRALS USING CYLINDRICAL AND SPHERICAL COORDINATES

CYLINDRICAL COORDINATES: Think: 'polar + z':



CONVERSION BETWEEN RECTANGULAR AND CYLINDRICAL COORDINATES:

$x = r\cos(\theta)$ $\qquad$ $y = r\sin(\theta)$ $\qquad$ $z = z$

$r^2 = x^2 + y^2$ $\qquad$ $\tan(\theta) = \dfrac{y}{x},\ x \neq 0$ $\qquad$ $z = z$

EXAMPLE 1: Convert $(3, \pi, 4)$ from cylindrical to rectangular coordinates.

Ans: $(-3, 0, 4)$

EXAMPLE 2: Convert $(-1, 2, 5)$ from rectangular to cylindrical coordinates.

Ans: $\left(\sqrt{5}, \pi - \tan^{-1}(2), 5\right)$

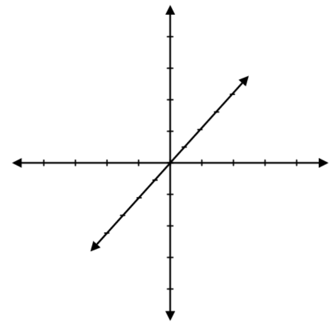
EXAMPLE 3: Convert the following equations from rectangular to cylindrical coordinates:

1. $z = 9 - x^2 - y^2$
2. $z = 6x$

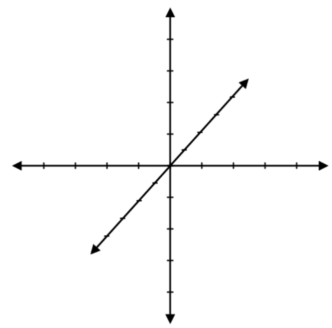
- Ans: $z = 9 - r^2$
- Ans: $z = 6r\cos(\theta)$

EXAMPLE 4: Sketch or otherwise describe the graphs of the followings equations in cylindrical coordinates:

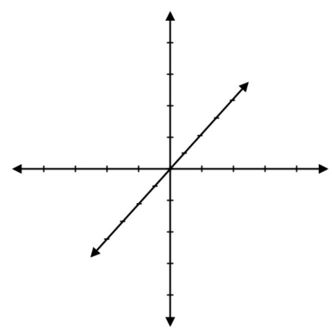
1. $r = 3$



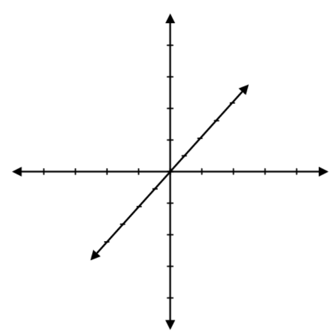
2. $\theta = \dfrac{\pi}{2}$



3. $z = 3$

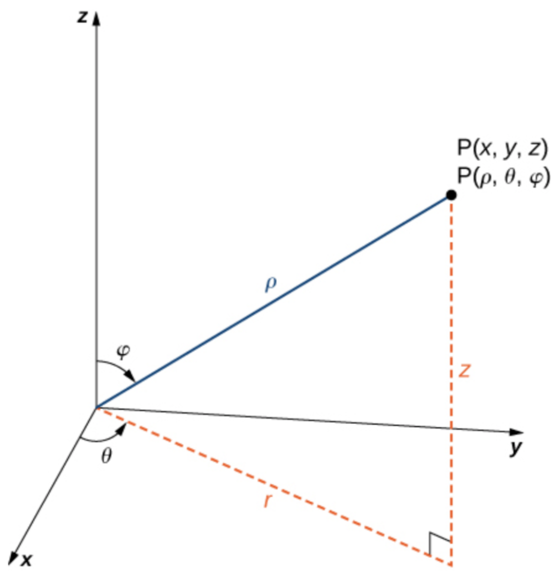


4. $z = r$



SPHERICAL COORDINATES: Think 'distance from the origin, longitude, latitude': (ρ, θ, ϕ)

NOTE: ρ and ϕ have the following restrictions: $\rho \geq 0$ and $0 \leq \phi \leq \pi$.



CONVERSION BETWEEN RECTANGULAR AND SPHERICAL COORDINATES:

$x = \rho \sin(\phi)\cos(\theta)$ $\qquad$ $y = \rho \sin(\phi)\sin(\theta)$ $\qquad$ $z = \rho\cos(\phi)$

$\rho^2 = x^2 + y^2 + z^2$ $\qquad$ $\tan(\theta) = \dfrac{y}{x},\ x \neq 0$ $\qquad$ $\cos(\phi) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

EXAMPLE 5: Convert $\left(4, \frac{\pi}{2}, \frac{\pi}{4}\right)$ from spherical to rectangular coordinates.

Ans: $(0, 2\sqrt{2}, 2\sqrt{2})$

EXAMPLE 6: Convert $(-1, -2, -3)$ from rectangular to spherical coordinates.

Ans: $\left(\sqrt{14}, \pi + \tan^{-1}(2), \cos^{-1}\left(-\frac{3}{\sqrt{14}}\right)\right)$

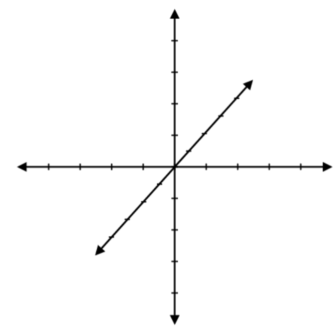
EXAMPLE 7: Convert the following equations from rectangular to spherical coordinates:

1. $z = -\sqrt{3x^2 + 3y^2}$
   - Ans: $\phi = \dfrac{5\pi}{6}$

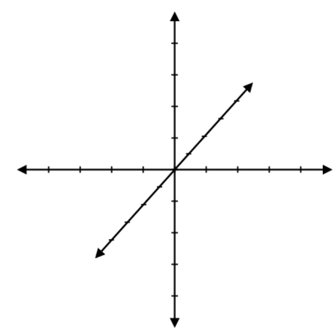
2. $x^2 - 4x + y^2 + z^2 = 0$
   - Ans: $\rho = 4\sin(\phi)\cos(\theta)$

EXAMPLE 8: Sketch or otherwise describe the graphs of the followings equations in spherical coordinates:

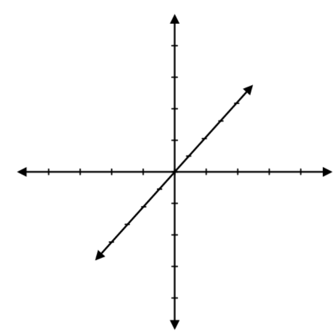
1. $\rho = 3$



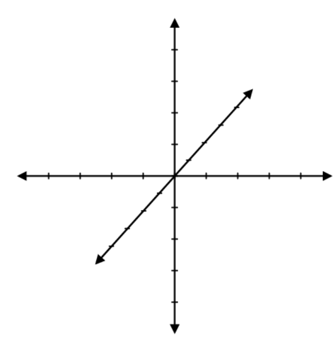
2. $\theta = \dfrac{\pi}{2}$



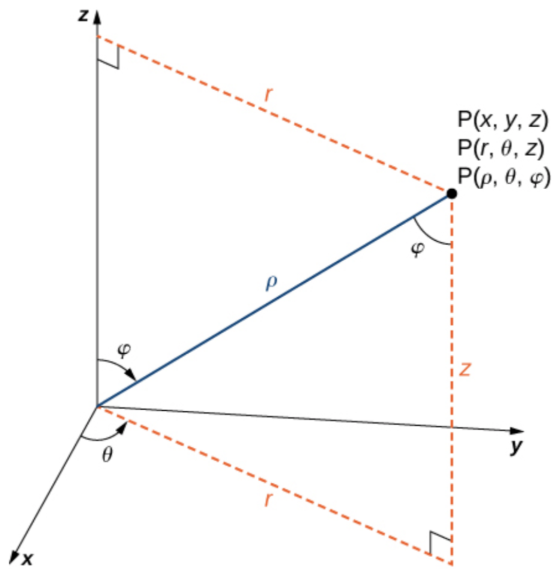
3. $\phi = \dfrac{3\pi}{4}$



4. $\rho = 4\cos(\phi)$



CONVERSION BETWEEN CYLINDRICAL AND SPHERICAL COORDINATES:



$r = \rho\sin(\phi)$	if $r \geq 0$, $\theta = \theta$	$z = \rho\cos(\phi)$
$\rho^2 = r^2 + z^2$	if $r \geq 0$, $\theta = \theta$	$\cos(\phi) = \dfrac{z}{\sqrt{r^2+z^2}}$

EXAMPLE 9: Convert the following rectangular equations to cylindrical and spherical coordinates.

1. $z = x^2 + y^2$

   - Ans: $z = r^2$, $\rho = \csc(\phi)\cot(\phi)$

2. $z = \sqrt{16 - x^2 - y^2}$

   - Ans: $z = \sqrt{16 - r^2}$, $\rho = 4$, $0 \leq \phi \leq \dfrac{\pi}{2}$.

1. $x^2 + y^2 = 25$

   - Ans: $r = 5$, $\rho = 5\csc(\phi)$.

2. $x + 2y + 3z = 6$

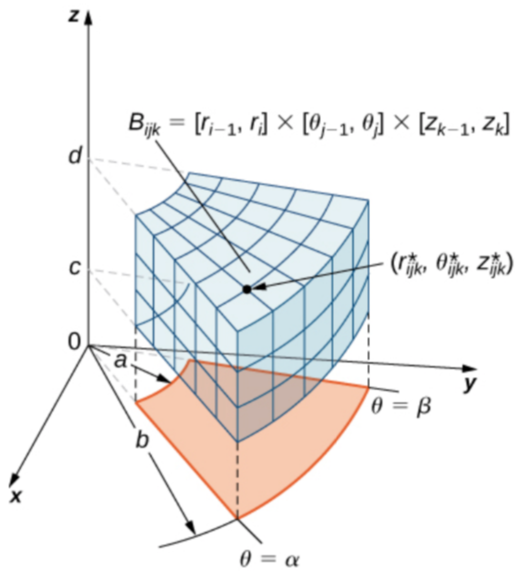
   - Ans: $r\cos(\theta) + 2r\sin(\theta) + 3z = 6$,
     $\rho = \dfrac{6}{\sin(\phi)\cos(\theta) + 2\sin(\phi)\sin(\theta) + 3\cos(\phi)}$

CYLINDRICAL COORDINATES: Suppose a solid Q can be described as

$$Q = \{(r, \theta, z) : \alpha \leq \theta \leq \beta,\ g_1(\theta) \leq r \leq g_2(\theta),\ h_1(r, \theta) \leq z \leq h_2(r, \theta)\}$$

then:

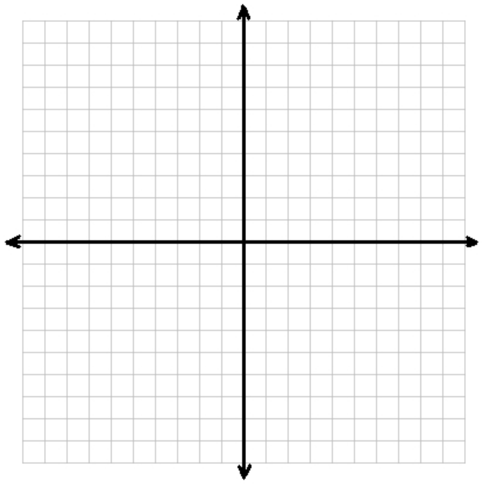
$$\iiint_Q f(x, y, z)\, dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z)\, dz\, r\, dr\, d\theta$$

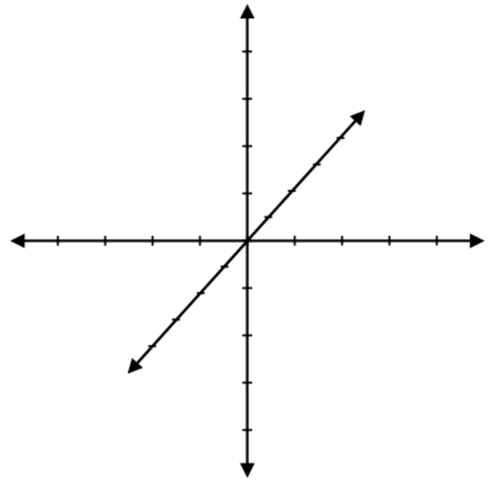


A typical volume element in cylindrical coordinates: $dV = dz\, r\, dr\, d\theta$

EXAMPLE 10: Sketch the solid over which each integral is being evaluated and convert to cylindrical coordinates.

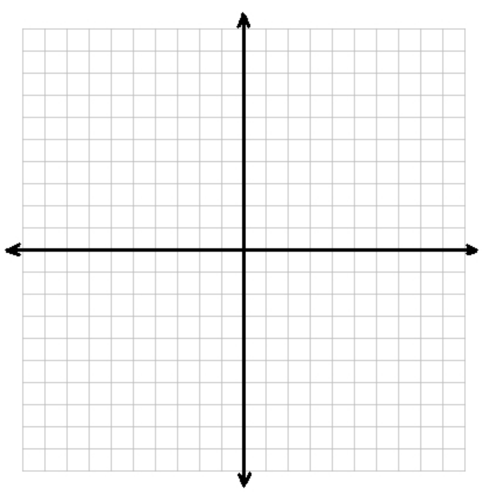
1. $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-2\sqrt{x^2+y^2}} x\, dz\, dy\, dx$

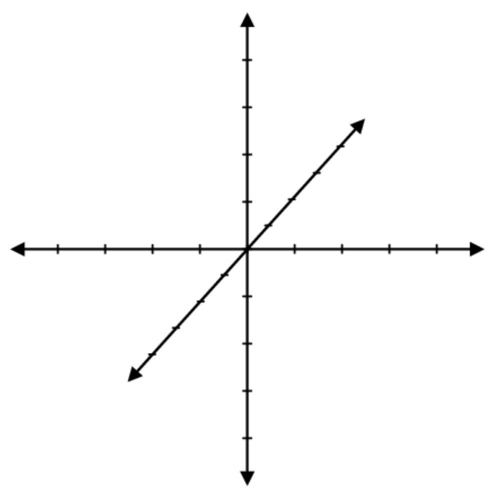




Ans: $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{4-2r} r^2 \cos(\theta)\, dz\, dr\, d\theta = \frac{16}{3}$

2. $\displaystyle\int_{-1}^1 \int_0^{1/2} \int_{y\sqrt{3}}^{\sqrt{1-y^2}} (x^2+y^2)^{1/2}\, dx\, dy\, dz$





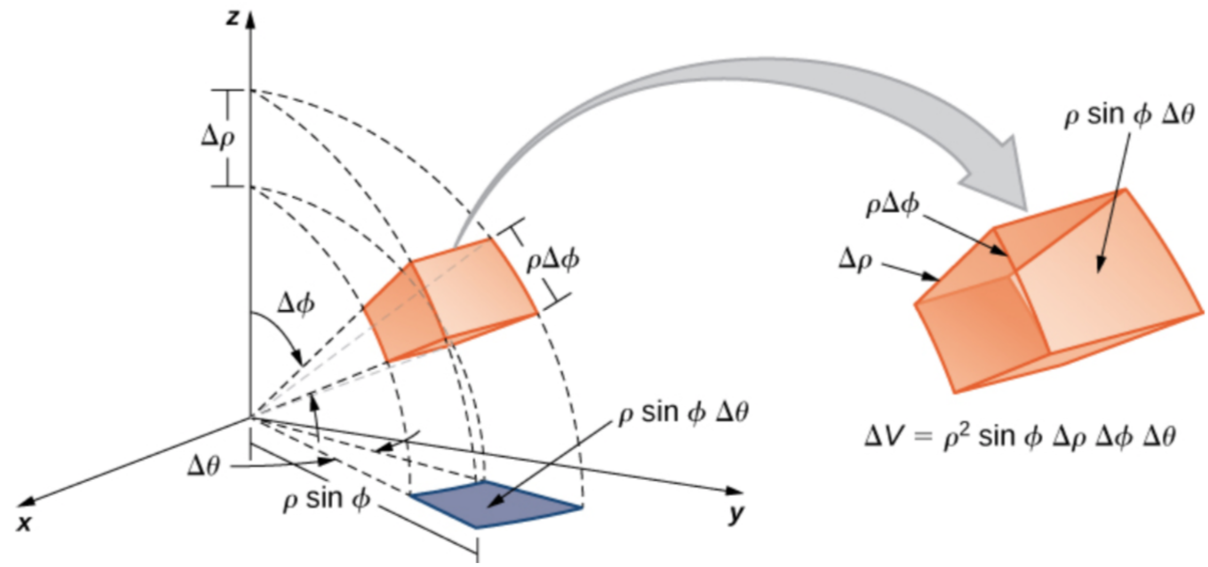
Ans: $\displaystyle\int_0^{\pi/6} \int_0^1 \int_{-1}^1 r^2\, dz\, dr\, d\theta = \frac{\pi}{9}$

SPHERICAL COORDINATES: Suppose a solid Q can be described as

$$Q = \{(\rho, \theta, \phi) : \alpha_1 \le \theta \le \alpha_2,\ \beta_1 \le \phi \le \beta_2,\ g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\}$$

then

$$\iiint_Q f(x, y, z)\, dV = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho \sin(\phi)\cos(\theta), \rho \sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$



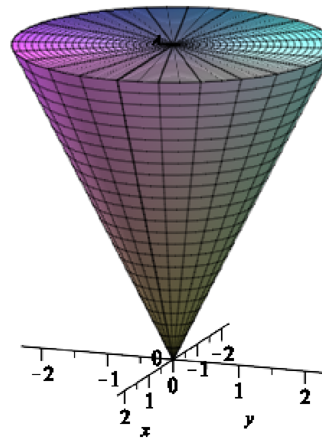
A typical volume element in cylindrical coordinates: $dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

EXAMPLE 11: Find the volume of the solid between $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$ using a triple iterated integral in spherical coordinates. Check your answer using a formula from geometry.

Ans: Volume $= \displaystyle\int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \frac{14\pi}{3}$ units^3

EXAMPLE 12: Find the volume of the indicated solid by evaluating an integral in spherical coordinates.

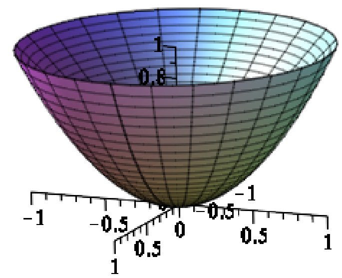
1. The portion of the cone bounded by $z = \sqrt{3x^2 + 3y^2}$ and $z = 4$.



Ans: $\displaystyle\int_0^{2\pi} \int_0^{\pi/6} \int_0^{4\sec(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \frac{64\pi}{9}\text{units}^3$

EXAMPLE 13: Find the volume of the indicated solid by evaluating an integral in spherical coordinates.

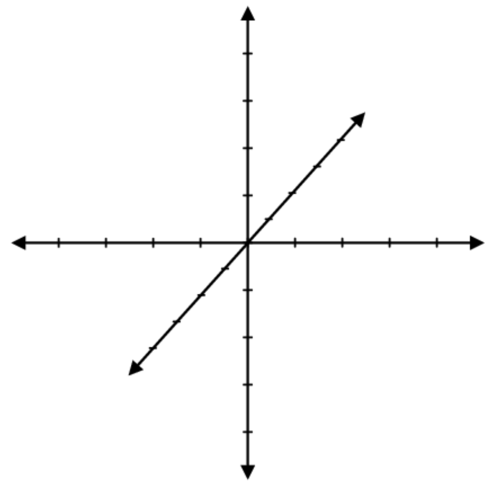
1. The solid bounded above $z = x^2 + y^2$ but below $z = \sqrt{x^2 + y^2}$.



Ans: $\displaystyle\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\csc(\phi)\cot(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \frac{\pi}{6}$ units^3

EXAMPLE 14: Let Q be the solid bounded between the two surfaces: $z = \sqrt{4 - x^2 - y^2}$ and $z = \dfrac{x^2 + y^2}{3}$.

1. Sketch or otherwise describe Q.



2. Set-up an integral in cylindrical coordinates which would compute the volume of Q.

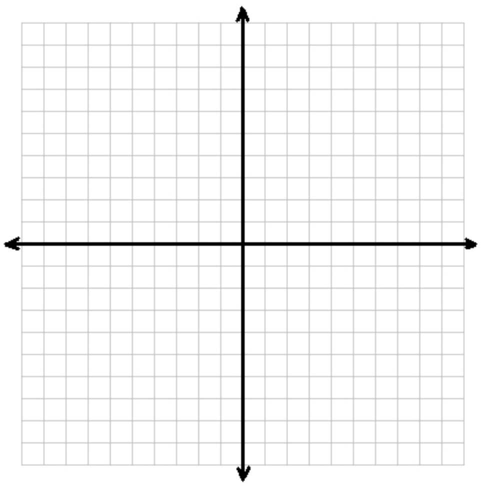
Ans: $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2/3}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = \frac{19\pi}{6}$ units^3

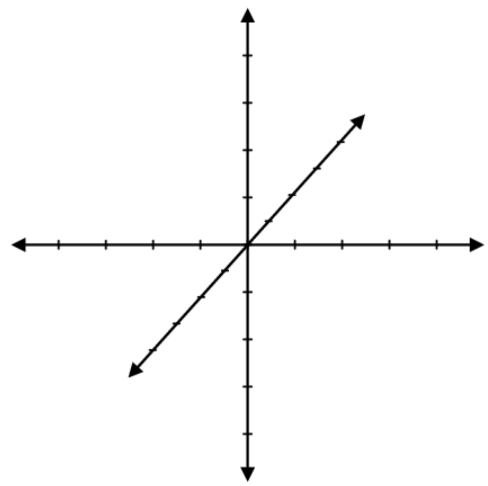
3. Set-up a sum of integrals in spherical coordinates which would compute the volume of Q.

Ans: $\displaystyle\int_0^{2\pi} \int_0^{\pi/3} \int_0^{2} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{3\csc(\phi)\cot(\phi)} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \frac{19\pi}{6}$ units^3

EXAMPLE 15: Sketch the solid over the integral is being evaluated and convert to spherical coordinates.

$$\int_{-\sqrt{12}}^{\sqrt{12}} \int_{-\sqrt{12-x^2}}^{\sqrt{12-x^2}} \int_{4-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} z\, dz\, dy\, dx$$





Ans: $\displaystyle\int_0^{2\pi} \int_0^{\pi/3} \int_0^{4} \rho^3 \cos(\phi)\sin(\phi)\, d\rho\, d\phi\, d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi} \int_0^{8\cos(\phi)} \rho^3 \cos(\phi)\sin(\phi)\, d\rho\, d\phi\, d\theta = \frac{160\pi}{3}$

HOMEWORK: Section 16.5: 11 - 55 every other odd; 63 - 75 every other odd.